Aggregation Kinetics and Network Evolution

Babak Fotouhi

McGill University

12 March 2012

Outline

- Introduction
- Basics of aggregation process
- Gelation
- Network Evolution

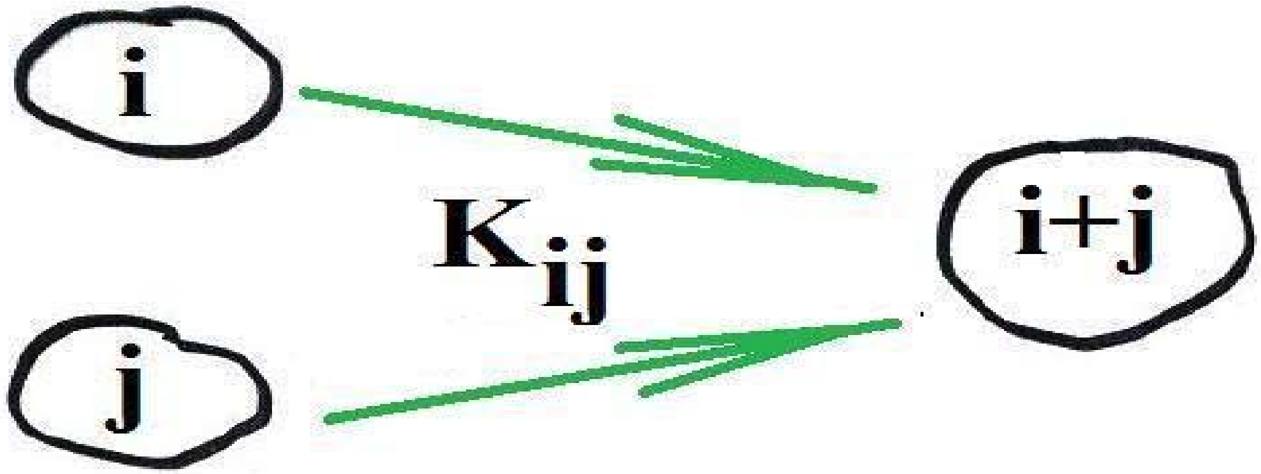
Introduction

- Clusters
- Examples:
  - Jello
  - Cheese
  - Stellar evolution
  - City population distribution
  - Wealth distribution
- With/without input
- Goal: $c_k(t)$

Introduction

- **Assumptions:**
  - **Spatial homogeneity (mean field)**
  - **Mass quanta (monomer, k-mer)**
  - **Bimolecular reactions**
  - **Shape independence**
  - **Initial condition: monomers**

Basics of Aggregation Process



$$A_i + A_j \xrightarrow{K_{ij}} A_{i+j}$$

Basics of Aggregation Process

The rate equation:

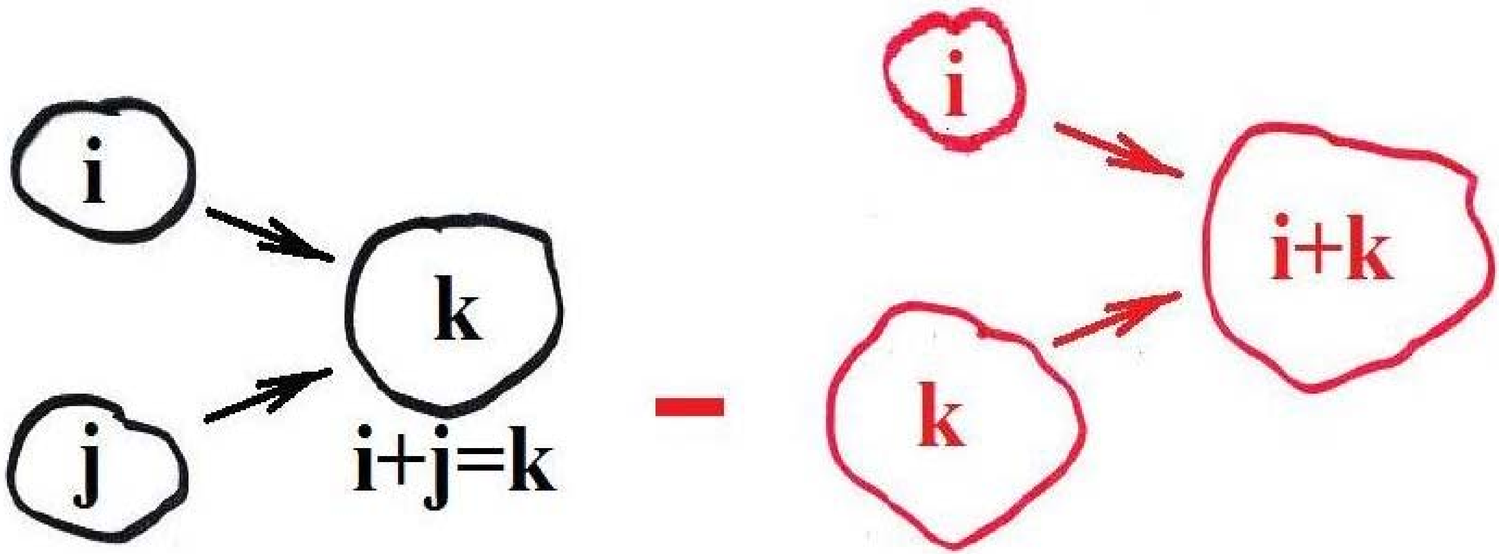
$$\frac{dc_k(t)}{dt} = \frac{1}{2}\sum_{i+j=k} K_{ij} c_i c_j - c_k \sum_{i \geq 1} K_{ik} c_i$$

$$M(t) \equiv \sum_k k c_k(t) \longrightarrow \dot{M}(t) = 0$$

$$\sum_k k c_k(t) \equiv 1$$

Basics of Aggregation Process

$$\frac{1}{2}\sum_{i+j=k} K_{ij} c_i c_j - c_k \sum_{i\geq 1} K_{ik} c_i$$



Basics of Aggregation Process

A simple case:

$$K_{ij} = const. \xrightarrow{S.E} 2 + (\frac{i}{j})^{1/3} + (\frac{j}{i})^{1/3} \approx 2$$

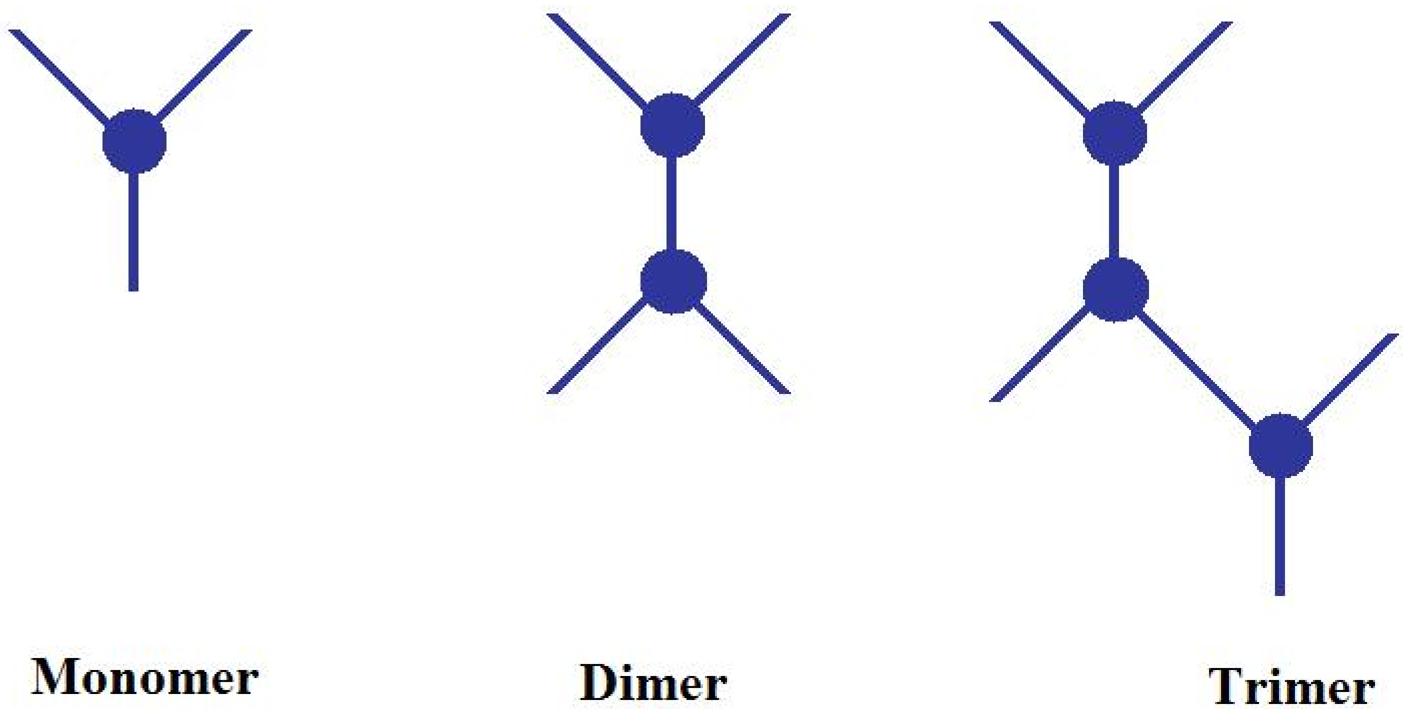
$$\frac{dc_k(t)}{dt} = \sum_{i+j=k} c_i c_j - 2c_k \sum_{i \geq 1} c_i$$

$$\rightarrow \quad c_k(t) = \frac{t^{k-1}}{(1+t)^{k+1}} \sim t^{-2} e^{-k/t}$$

Moments converge

Gelation

- **Product kernel:** $K_{ij} = i \, . \, j$
- **Rich gets richer**



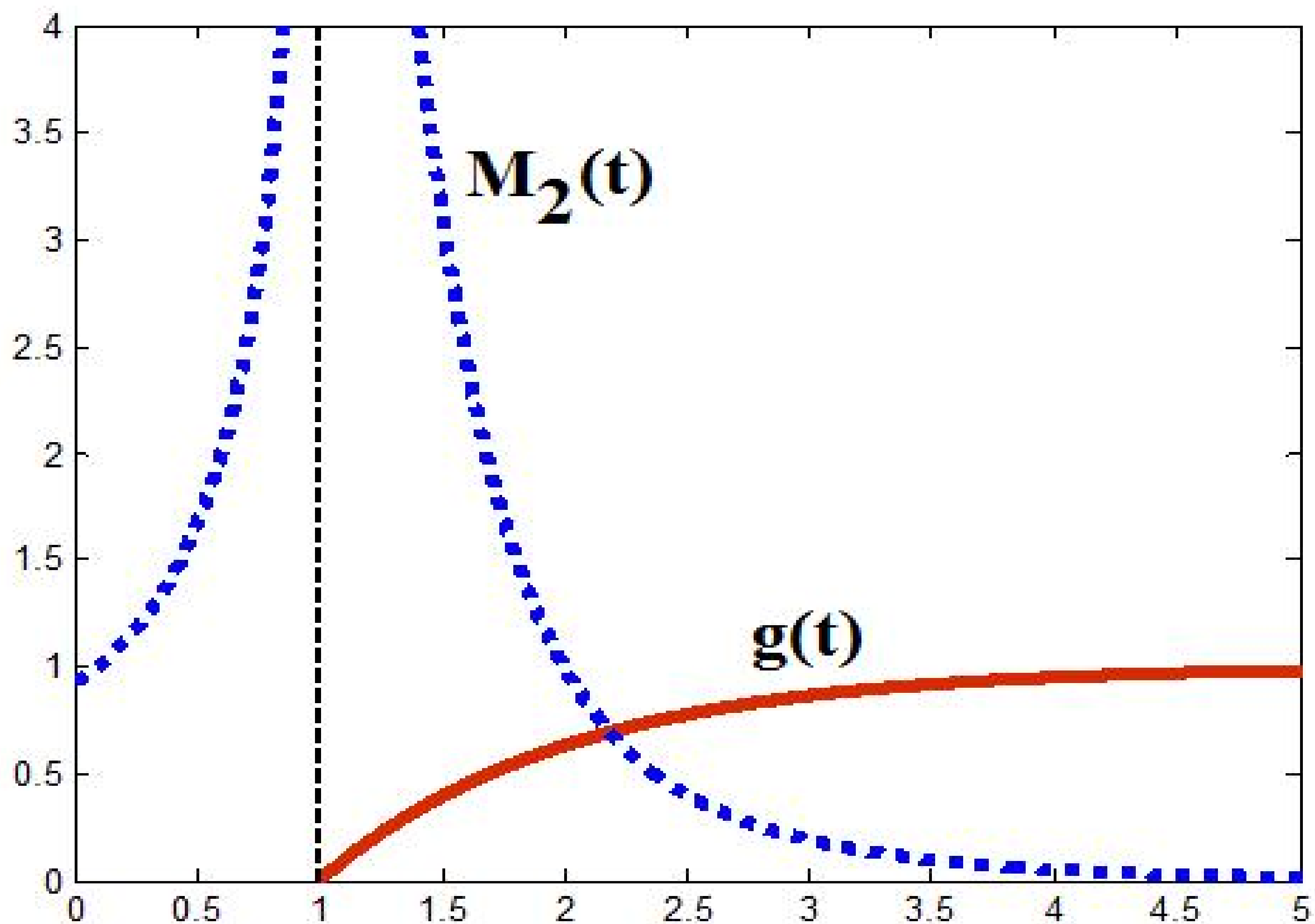
Gelation

- **Gel cluster:**
  - **Mass=g.M**
  - **Concentration=1/M**

$$c_k(t) \sim k^{-5/2} e^{\frac{-k(t-1)^2}{2}}$$

- **Second moment diverges**

Gelation



Network Evolution: Erdos-Renyi

- **Methods:**
  - **N nodes, link each pair with =p/N**
  - **Link “L” out of N(N-1)/2 pairs**
- **Kinetic formulation:**

$$\frac{dn_k}{dt} = n_{k-1} - n_k$$

$$\longrightarrow n_k(t) = \frac{t^k}{k!} e^{-t}$$

Network Evolution: Erdos-Renyi

- **Cluster size distribution**
- **N isolated clusters**

$$C_i + C_j \xrightarrow{(i\frac{C_i}{N})\times(j\frac{C_j}{N})} C_{i+j}$$

- **Aggregation, product kernel**
- **Jello = giant component**

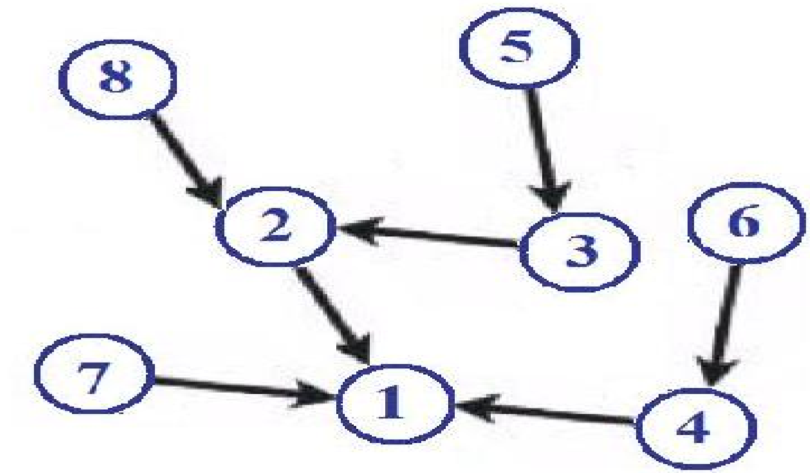
NETWORK EVOLUTION: RANDOM RECURSIVE TREE (RRT)

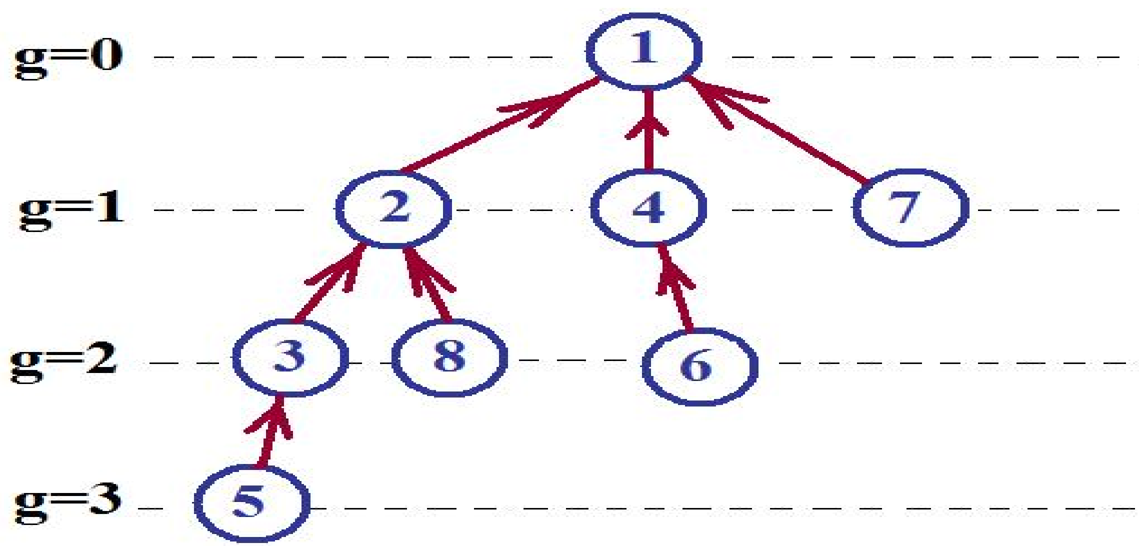
Network Evolution: RRT

$$\frac{dN_k}{dN} = \frac{N_{k-1}}{N} - \frac{N_k}{N} + \delta_{k,1}$$

$$\longrightarrow \quad n_k(t) = 2^{-k}$$

Network Evolution : RRT: Geneology





Network Evolution: RRT: Diameter

- **N nodes.**
  - **g = ?**
  - **L_g (N) = ?**
- **When does L_g increase?**

$$\frac{dL_g}{dN} = \frac{L_{g-1}}{N} \longrightarrow L_g(N) = \frac{(\ln N)^g}{g!}$$

Network Evolution: RRT: Diameter

$$L_g(N) = \frac{(\ln N)^g}{g!}$$

- **Grows with g up to** $g \sim \ln N$
- **Shrinks to O(1) for** $g \sim e \ln N$
- **Diameter** $\sim 2e \ln N$

Network Evolution: Preferential Attachment

$$\frac{dN_k}{dN} = \frac{A_{k-1}}{A} N_{k-1} - \frac{A_k}{A} N_k + \delta_{k,1}$$

$$\begin{cases} A_k = k^{\gamma} \\ \gamma >=< 1 \end{cases}$$

$$\gamma = 1 \longrightarrow n_k \sim \frac{\Gamma(k)}{\Gamma(k+3)} \sim k^{-3}$$

Summary

- Aggregation kinetics: Powerful
- Erdos-Renyi & Gelation
- RRT: Diameter, Distribution
- Preferential attachment: Distribution

References

- Ben-Naim, Krapivsky, Phys. Rev. E 68, 031104 (2003). arXiv: cond-mat/0305154
- Ben-Naim, Krapivsky, Phys. Rev. E 71, 026129 (2005). arXiv: cond-mat/0408620
- P. L. Krapivsky, S. Redner and E. Ben-Naim, “A Kinetic View of Statistical Physics” (Cambridge University Press, Cambridge, 2010)

Thank you